

Gorenstein Homological Theory for Differential Modules *

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Abstract

We show that a differential module is Gorenstein projective if and only if its underlying module is Gorenstein projective. Dually, a differential module is Gorenstein injective if and only if its underlying module is Gorenstein injective.

1 Introduction

Let R be a ring. A differential module (M, δ) over R is by definition an R -module M equipped with an R -endomorphism δ of square zero. A homomorphism from a differential module (M, δ_M) to a differential module (N, δ_N) is an R -homomorphism from M to N such that $\delta_M f = f \delta_N$. We denote by $\mathbf{Diff}(R)$ the category of all differential R -modules. Note that it is just the module category of the ring of dual numbers over R [6]. In particular, it admits enough projective objects and enough injective objects.

The notion of differential modules already appeared in Cartan and Eilenberg's book [6] five decades ago. However, it is recent thing that the study of differential modules is founded interesting in their own right. In the paper [2], Avramov etc. studied class and rank of differential modules, given common generalizations of important results in commutative algebra and graded polynomial rings. Ringel and Zhang [11] recently provided interesting relationship between the module category of a hereditary (artin) algebra R and the stable category of Frobenius category \mathcal{L} of perfect differential modules over R , and they also described the representation theory of \mathcal{L} .

This paper focuses on Gorenstein homological theory of differential modules. Gorenstein homological theory originated in the works of Auslander and Bridger [1], where they introduced G-dimensions. Enochs extended their ideas and introduced Gorenstein projective, Gorenstein injective and Gorenstein flat modules and correspondent dimensions over arbitrary rings, see the book [9] for details. Later Gorenstein homological theory was extensively studied and developed by Avramov, Martsinkovsky, Christensen, Veliche, Sather-Wagstaff, Chen, Beligiannis, Yang and many others (see for instance [3, 4, 7, 8, 10, 12, 13, 14] etc.).

Our main result states as follows.

Theorem 1.1 *Let R be a ring and (M, δ) a differential module. Then (M, δ) is Gorenstein projective (in the category of differential modules) if and only if M is Gorenstein projective (in*

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the category of R -modules). Dually, (M, δ) is Gorenstein injective if and only if M is Gorenstein injective.

The theorem provides interesting relationships between Gorenstein homology theory of differential modules and that of their underlying modules. Furthermore, we have the following result. Here, $\text{Gpd}M$ (respectively, $\text{Gid}M$) means the Gorenstein projective (respectively, Gorenstein injective) dimension of the module M and $\text{Ggd}R$ means the Gorenstein global dimension of R .

Theorem 1.2 *Let (M, δ_M) be a differential module and n be an integer.*

- (1) $\text{Gpd}(M, \delta_M) \leq n$ if and only if $\text{Gpd}M \leq n$.
- (2) $\text{Gid}(M, \delta_M) \leq n$ if and only if $\text{Gid}M \leq n$.
- (3) $\text{GgdDiff}(\mathbf{R}) \leq n$ if and only if $\text{Ggd}R \leq n$.

2 Basic results on differential modules

We write the element in a direct sum as a row vector. For two maps $f : M \rightarrow N$ and $g : N \rightarrow L$, we use fg to denote the composition of these two maps. Throughout the paper, we fix R a ring (which is associative with an identity).

Let X be an R -module. It is easy to see that $(X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$ is a differential module and such differential module is called contractible [2].

The following easy lemma is useful in describing homomorphisms between a differential module and a contractible differential module. We leave its proof to the reader.

Lemma 2.1 *Let (M, δ) be a differential module and X an R -module.*

- (1) *Let $f \in \text{Hom}_R(M, X \oplus X)$. Then $f \in \text{Hom}_{\text{Diff}(\mathbf{R})}((M, \delta), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ if and only if $f = (g \quad \delta g)$ for some $g \in \text{Hom}_R(M, X)$.*
- (2) *Let $f \in \text{Hom}_R(X \oplus X, M)$. Then $f \in \text{Hom}_{\text{Diff}(\mathbf{R})}((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), (M, \delta))$ if and only if $f = \begin{pmatrix} g\delta \\ g \end{pmatrix}$ for some $g \in \text{Hom}_R(X, M)$.*

By the above lemma, we can represent every homomorphism from the differential module (M, δ) to a contractible differential module $(X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$ as the form $(g \quad \delta g)$, where $g \in \text{Hom}_R(M, X)$. Similarly, we represent every homomorphism from $(X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$ to (M, δ) as the form $\begin{pmatrix} g\delta \\ g \end{pmatrix}$.

Lemma 2.2 *Let $f : (M, \delta_M) \rightarrow (N, \delta_N)$ be a homomorphism between differential modules and X an R -module.*

- (1) *$\text{Hom}_R(f, X)$ is an epimorphism if and only if $\text{Hom}_{\text{Diff}(\mathbf{R})}(f, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is an epimorphism.*
- (2) *$\text{Hom}_R(X, f)$ is an epimorphism if and only if $\text{Hom}_{\text{Diff}(\mathbf{R})}((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), f)$ is an epimorphism.*

Proof. (1) Assume first $\text{Hom}_R(f, X)$ is an epimorphism. Given a homomorphism $(g \ \delta_M g) \in \text{Hom}_{\mathbf{Diff}(\mathbf{R})}((M, \delta_M), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$, where $g \in \text{Hom}_R(M, X)$, then we have that $g = fh$ for some $h \in \text{Hom}_R(N, X)$, by assumptions. Since $\delta_M f = f\delta_N$, we see that $(g \ \delta_M g) = f(h \ \delta_N h)$ with $(h \ \delta_N h) \in \text{Hom}_{\mathbf{Diff}(\mathbf{R})}((N, \delta_N), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$. It follows that $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(f, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is an epimorphism.

Assume now $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(f, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is an epimorphism. Given a homomorphism $g \in \text{Hom}_R(M, X)$, we have $(g \ \delta_M g) \in \text{Hom}_{\mathbf{Diff}(\mathbf{R})}((M, \delta_M), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$. Hence there is some $(h \ \delta_N h) \in \text{Hom}_{\mathbf{Diff}(\mathbf{R})}((N, \delta_N), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ such that $(g \ \delta_M g) = f(h \ \delta_N h)$, where $h \in \text{Hom}_R(N, X)$. It follows that $g = fh$ and so $\text{Hom}_R(f, X)$ is an epimorphism.

(2) The proof is dual to that of part (1). \square

We have the following useful corollary.

Corollary 2.3 *Assume that $0 \rightarrow (M, \delta_M) \rightarrow (N, \delta_N) \rightarrow (L, \delta_L) \rightarrow 0$ (\dagger) is an exact sequence of differential modules and X is an R -module. Then*

(1) $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\dagger, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact if and only if $\text{Hom}_R(\dagger, X)$ is exact.

(2) $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \dagger)$ is exact if and only if $\text{Hom}_R(X, \dagger)$ is exact.

3 Gorenstein differential modules

We begin with the following general result.

Lemma 3.1 *Let (M, δ) be a differential module and \mathcal{C} be a class of R -modules. Assume that*

$$0 \rightarrow L \xrightarrow{\lambda} C \xrightarrow{\pi} M \rightarrow 0 \quad (\ddagger)$$

is an exact sequence of R -modules such that $C \in \mathcal{C}$ and $\text{Hom}_R(C', \ddagger)$ is exact for any $C' \in \mathcal{C}$. Then there is an exact sequence of differential modules

$$0 \rightarrow (C \oplus L, \delta_{C \oplus L}) \xrightarrow{\begin{pmatrix} -1 & h \\ 0 & \lambda \end{pmatrix}} (C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{\begin{pmatrix} \pi\delta \\ \pi \end{pmatrix}} (M, \delta) \rightarrow 0, \quad (\ddagger\ddagger)$$

for some $h \in \text{Hom}_R(C, C)$, such that

(1) $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((C' \oplus C', \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \ddagger\ddagger)$ is exact for any $C' \in \mathcal{C}$.

(2) *For any R -module X , $\text{Hom}_R(\ddagger, X)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\ddagger\ddagger, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact.*

Proof. Since $\text{Hom}_R(C, \ddagger)$ is exact by assumptions, $\text{Hom}_R(C, \pi)$ is epi. Hence, there is some $h \in \text{Hom}_R(C, C)$ such that $\pi\delta = h\pi$. Now we can construct the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \rightarrow & L & \xrightarrow{(0 \ 1)} & C \oplus L & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & C \rightarrow 0 \\
& & \parallel & & \begin{pmatrix} h \\ \lambda \end{pmatrix} \downarrow & \swarrow h & \downarrow \pi\delta \\
0 & \rightarrow & L & \xrightarrow{\lambda} & C & \xrightarrow{\pi} & M \rightarrow 0
\end{array}$$

It is easy to see that the above diagram is a pullback. Hence there is an exact sequence

$$0 \rightarrow C \oplus L \xrightarrow{\begin{pmatrix} -1 & h \\ 0 & \lambda \end{pmatrix}} C \oplus C \xrightarrow{\begin{pmatrix} \pi\delta \\ \pi \end{pmatrix}} M \rightarrow 0.$$

Note that the homomorphism $\begin{pmatrix} \pi\delta \\ \pi \end{pmatrix} \in \text{Hom}_{\mathbf{Diff}(\mathbf{R})}((C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), (M, \delta))$, so we indeed have an exact sequence of differential modules.

$$0 \rightarrow (C \oplus L, \delta_{C \oplus L}) \xrightarrow{\begin{pmatrix} -1 & h \\ 0 & \lambda \end{pmatrix}} (C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{\begin{pmatrix} \pi\delta \\ \pi \end{pmatrix}} (M, \delta) \rightarrow 0, \quad (\ddagger\ddagger)$$

For any $C' \in \mathcal{C}$, we have that $\text{Hom}_R(C', \pi)$ is epi, since $\text{Hom}_R(C', \ddagger)$ is exact by assumptions. It follows easily that $\text{Hom}_R(C', \begin{pmatrix} \pi\delta \\ \pi \end{pmatrix})$ is also epi., which in turn means that $\text{Hom}_R(C', \ddagger\ddagger)$ is exact. Hence we obtain that $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \ddagger\ddagger)$ is exact for any $C' \in \mathcal{C}$, by Corollary 2.3.

Now take any R -module X . On one hand, suppose that $\text{Hom}_R(\ddagger, X)$ is exact. Then, given a homomorphism $\begin{pmatrix} g_C \\ g_L \end{pmatrix} \in \text{Hom}_R(C \oplus L, X)$, there is some $\theta \in \text{Hom}_R(C, X)$ such that $g_L = \lambda\theta$. Then we can check that $\begin{pmatrix} g_C \\ g_L \end{pmatrix} = \begin{pmatrix} -1 & h \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} h\theta - g_C \\ \theta \end{pmatrix}$. Hence $\text{Hom}_R(\ddagger\ddagger, X)$ is exact. Consequently, $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\ddagger\ddagger, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact by Corollary 2.3.

On the other hand, suppose that $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\ddagger\ddagger, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact, which is equivalent to say that $\text{Hom}_R(\ddagger\ddagger, X)$ is exact by Corollary 2.3. Given a homomorphism $x \in \text{Hom}_R(L, X)$, we obtain $\begin{pmatrix} 0 \\ x \end{pmatrix} \in \text{Hom}_R(C \oplus L, X)$. It follows that there is some $\begin{pmatrix} y \\ z \end{pmatrix} \in \text{Hom}_R(C \oplus C, X)$ such that $\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} -1 & h \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$. Hence $x = \lambda z$ and consequently $\text{Hom}_R(\lambda, X)$ is epi. Thus, $\text{Hom}_R(\ddagger, X)$ is exact. \square

Dually, we have the following result.

Lemma 3.2 *Let (M, δ) be a differential module and \mathcal{C} be a class of R -modules. Assume that*

$$0 \rightarrow M \xrightarrow{\lambda} C \xrightarrow{\pi} L \rightarrow 0 \quad \ddagger^o$$

is an exact sequence of R -modules such that $C \in \mathcal{C}$ and $\text{Hom}_R(\ddagger^o, C')$ is exact for any $C' \in \mathcal{C}$. Then there is an exact sequence of differential modules

$$0 \rightarrow (M, \delta) \xrightarrow{(\lambda \ \delta\lambda)} (C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{\begin{pmatrix} \pi & h \\ 0 & -1 \end{pmatrix}} (L \oplus C, \delta_{L \oplus C}) \rightarrow 0, \quad (\ddagger\ddagger^o)$$

for some $h \in \text{Hom}_R(C, C)$, such that

$$(1) \text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\ddagger^o, (C' \oplus C', \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})) \text{ is exact for any } C' \in \mathcal{C}.$$

(2) For any R -module X , $\text{Hom}_R(X, \dagger^o)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \dagger \dagger^o)$ is exact.

Let \mathcal{C} be a class of R -modules. Recall that a proper \mathcal{C} -resolution of an R -module M is an exact sequence $\cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$ (\star), where $C_i \in \mathcal{C}$ for all $i \geq 0$, such that $\text{Hom}_R(C, \star)$ is exact for any $C \in \mathcal{C}$.

Dually, a proper \mathcal{C} -coresolution of an R -module M is an exact sequence $0 \rightarrow M \rightarrow C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots$ (\star^o), where $C_i \in \mathcal{C}$ for all $i \geq 0$, such that $\text{Hom}_R(\star^o, C)$ is exact for any $C \in \mathcal{C}$.

In the following, we denote by \mathcal{C}^δ the class of differential modules $(C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$ with $C \in \mathcal{C}$.

Lemma 3.3 Let $(M, \delta_M) \in \mathbf{Diff}(\mathbf{R})$ and \mathcal{C} be a class of R -modules which is closed under direct sums. Assume that there is a proper \mathcal{C} -resolution of the R -module M , say,

$$\cdots \rightarrow C_2 \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \xrightarrow{c_0} M \rightarrow 0 \quad (\natural)$$

Denote $M_i = \text{Im} c_i$ for $i \geq 0$. Then there is a proper \mathcal{C}^δ -resolution of the differential module (M, δ_M)

$$\cdots \rightarrow (Q_2 \oplus Q_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_2} (Q_1 \oplus Q_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_1} (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_0} (M, \delta_M) \rightarrow 0 \quad (\natural_0)$$

such that

(1) $Q_i \simeq \bigoplus_{k=0}^i C_k$ and $\text{Ker} q_i \simeq Q_i \oplus M_{i+1}$, for all $i \geq 0$;

(2) For any R -module X , $\text{Hom}_R(\natural, X)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\natural_0, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact.

Proof. Note that $M_0 = M$ and we have exact sequences

$$0 \rightarrow M_{i+1} \xrightarrow{\lambda_i} C_i \xrightarrow{\pi_i} M_i \rightarrow 0 \quad (\natural_i)$$

such that $\pi_i \lambda_{i-1} = c_i$ and $\text{Hom}_R(C, \natural_i)$ is exact for any $C \in \mathcal{C}$.

Consider firstly the exact sequence

$$0 \rightarrow M_1 \xrightarrow{\lambda_0} C_0 \xrightarrow{\pi_0} M_0 \rightarrow 0. \quad (\natural_0)$$

By Lemma 3.1, we obtain an exact sequences of differential modules

$$0 \rightarrow (C_0 \oplus M_1, \delta_{C_0 \oplus M_1}) \rightarrow (C_0 \oplus C_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow (M_0, \delta_{M_0}) \rightarrow 0, \quad (\natural_0)$$

such that $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \natural_0)$ is exact for any $(C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \in \mathcal{C}^\delta$ and such that

$\text{Hom}_R(\natural_0, X)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\natural_0, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact, for any R -module X .

Note that we have an exact sequence of R -modules

$$0 \rightarrow M_2 \xrightarrow{\begin{pmatrix} 0 & \lambda_1 \end{pmatrix}} C_0 \oplus C_1 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \pi_1 \end{pmatrix}} C_0 \oplus M_1 \rightarrow 0 \quad (\natural'_1)$$

which comes from (\natural_1) and that $\text{Hom}_R(C, \natural'_1)$ is exact for any $C \in \mathcal{C}$, since $\text{Hom}_R(C, \natural_1)$ is exact. Thus, by repeating the above process to the exact sequence (\natural'_1) instead of (\natural_0) , and so on, we obtain exact sequences of differential modules

$$0 \rightarrow (N_{i+1}, \delta_{N_{i+1}}) \rightarrow (Q_i \oplus Q_i, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow (N_i, \delta_{N_i}) \rightarrow 0, \quad (\natural_i)$$

where $Q_i \simeq \oplus_{k=0}^i C_k$ and $N_{i+1} \simeq Q_i \oplus M_{i+1}$, such that $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \natural_i)$ is exact for any $(C \oplus C, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \in \mathcal{C}^\delta$ and such that $\text{Hom}_R(\natural'_i, X)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\natural_i, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact, for any R -module X .

Now the desired sequence (\natural_i) follows by combining together the short exact sequences (\natural_i) 's. \square

We also have the following dual result.

Lemma 3.4 *Let $(M, \delta_M) \in \mathbf{Diff}(\mathbf{R})$ and \mathcal{C} be a class of R -modules which is closed under direct sums. Assume that there is a proper \mathcal{C} -coresolution of the R -module M , say,*

$$0 \rightarrow M \xrightarrow{c_0} C_0 \xrightarrow{c_1} C_1 \xrightarrow{c_2} C_2 \rightarrow \cdots \quad (\natural^o)$$

Denote $M_i = \text{Im} c_i$ for $i \geq 0$. Then there is a proper \mathcal{C}^δ -coresolution of the differential module (M, δ_M)

$$0 \rightarrow (M, \delta) \xrightarrow{q_0} (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_1} (Q_1 \oplus Q_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_2} (Q_2 \oplus Q_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow \cdots \quad (\natural^o)$$

such that

(1) $Q_i \simeq \oplus_{k=0}^i C_k$ and $\text{Coker} q_i \simeq M_{i+1} \oplus Q_i$, for all $i \geq 0$;

(2) For any R -module X , $\text{Hom}_R(X, \natural^o)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \natural^o)$ is exact.

The following result describes differential modules which are orthogonal to contractible differential modules.

Proposition 3.5 *Let $(M, \delta) \in \mathbf{Diff}(\mathbf{R})$ and X be an R -module. Then*

(1) $\text{Ext}_{\mathbf{Diff}(\mathbf{R})}^i((M, \delta), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})) = 0$ for all $i \geq 1$ if and only if $\text{Ext}_R^i(M, X) = 0$ for all $i \geq 0$.

(2) $\text{Ext}_{\mathbf{Diff}(\mathbf{R})}^i((X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), (M, \delta)) = 0$ for all $i \geq 1$ if and only if $\text{Ext}_R^i(X, M) = 0$ for all $i \geq 0$.

Proof. (1) Take a projective resolution of the R -module M

$$\cdots \rightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \rightarrow 0. \quad (\natural_P)$$

By Lemma 3.3, there is a projective resolution of the differential module (M, δ_M)

$$\cdots \rightarrow (Q_2 \oplus Q_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_2} (Q_1 \oplus Q_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_1} (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_0} (M, \delta_M) \rightarrow 0, \quad (\natural_P)$$

such that $\text{Hom}_R(\natural_P, X)$ is exact if and only if $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\natural_P, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact. Hence, we can deduce that

$$\text{Ext}_{\mathbf{Diff}(\mathbf{R})}^i((M, \delta), (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})) = 0 \text{ for all } i \geq 1$$

$$\begin{aligned}
&\Leftrightarrow \text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\natural_P, (X \oplus X, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})) \text{ is exact} \\
&\Leftrightarrow \text{Hom}_R(\natural_P, X) \text{ is exact} \\
&\Leftrightarrow \text{Ext}_R^i(M, X) = 0 \text{ for all } i \geq 1.
\end{aligned}$$

(2). The proof is dual to that of (1). \square

Given a ring R , a Gorenstein projective module G is defined to be the image of the homomorphism g_0 which is given by an exact sequence of projective R -modules

$$\cdots \rightarrow P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} P_{-1} \xrightarrow{g_{-1}} \cdots \quad (\dagger^G)$$

such that $\text{Hom}_R(\dagger^G, P)$ is exact for any projective R -module P .

Dually, a Gorenstein injective module Q is defined to be the image of the homomorphism q_0 which is given by an exact sequence of injective R -modules

$$\cdots \rightarrow I_1 \xrightarrow{q_1} I_0 \xrightarrow{q_0} I_{-1} \xrightarrow{q_{-1}} \cdots \quad (\dagger^Q)$$

such that $\text{Hom}_R(I, \dagger^Q)$ is exact for any injective R -module I .

Note that the projective differential module over R , i.e., the projective object in $\mathbf{Diff}(\mathbf{R})$, are just contractible differential modules such that their underlying modules are projective R -modules [6, Chap. IV]. Similarly, the injective differential module over R , i.e., the injective object in $\mathbf{Diff}(\mathbf{R})$, are just contractible differential modules such that their underlying modules are injective R -modules.

We now describe Gorenstein projective (respectively, Gorenstein injective) differential modules over R .

Theorem 3.6 *Let (M, δ_M) be a differential module.*

- (1) *(M, δ_M) is Gorenstein projective if and only if M is a Gorenstein projective R -module.*
- (2) *(M, δ_M) is Gorenstein injective if and only if M is a Gorenstein injective R -module.*

Proof. (1) The only-if-part. Since (M, δ_M) is Gorenstein projective, there is an exact sequence of projective differential modules

$$\cdots \rightarrow (P_1, \delta_1) \rightarrow (P_0, \delta_0) \xrightarrow{f} (P_{-1}, \delta_{-1}) \rightarrow \cdots \quad (\dagger^G)$$

such that $(M, \delta_M) = \text{Im} f$ and $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(\dagger^G, (P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact for any projective differential module $(P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$. It follows from Corollary 2.3 that $\text{Hom}_R(\dagger^G, P)$ is also exact for any projective module P . Hence we see that M is Gorenstein projective.

The if-part. Assume that M is Gorenstein projective, then there is an exact sequence of projective modules

$$\cdots \rightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} P_{-1} \xrightarrow{p_{-1}} P_{-2} \xrightarrow{p_{-2}} \cdots \quad (*)$$

such that $M = \text{Im} p_0$ and $\text{Hom}_R(*, P)$ is exact for any projective module P . Then we have two exact sequences

$$\cdots \rightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{\pi_0} M \rightarrow 0 \quad (*_l)$$

and

$$0 \rightarrow M \xrightarrow{\lambda_0} P_{-1} \xrightarrow{p_{-1}} P_{-2} \xrightarrow{p_{-2}} \cdots, \quad (*_r)$$

such that both $\text{Hom}_R(*_l, P)$ and $\text{Hom}_R(*_r, P)$ are exact for any projective R -module P .

By Lemmas 3.3 and 3.4, we obtain two exact sequences

$$\cdots \rightarrow (Q_2 \oplus Q_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_2} (Q_1 \oplus Q_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_1} (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{\pi'_0} (M, \delta_M) \rightarrow 0 \quad (**_l)$$

and

$$0 \rightarrow (M, \delta_M) \xrightarrow{\lambda_0'} (Q_{-1} \oplus Q_{-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_{-1}'} (Q_{-2} \oplus Q_{-2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_{-2}'} \cdots, \quad (**_r)$$

where Q_i 's are projective R -modules, such that both $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(**_l, (P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ and $\text{Hom}_R(**_r, (P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ are exact for any projective differential module $(P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$.

Now combining sequences $(**_l)$ and $(**_r)$ together we obtain an exact sequence of projective differential modules

$$\begin{aligned} \cdots \rightarrow (Q_2 \oplus Q_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_2} (Q_1 \oplus Q_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_1} (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_0} \\ (Q_{-1} \oplus Q_{-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_{-1}'} (Q_{-2} \oplus Q_{-2}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \xrightarrow{q_{-2}'} \cdots, \end{aligned} \quad (**)$$

such that $(M, \delta_M) = \text{Im} q_0$ and $\text{Hom}_{\mathbf{Diff}(\mathbf{R})}(**, (P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}))$ is exact for any projective differential module $(P \oplus P, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$. It follows that (M, δ_M) is a Gorenstein projective differential module.

(2) The proof is dual to that of (1). \square

Let M be an R -module. The Gorenstein projective dimension of M , denoted by $\text{Gpd}M$, is defined to be the minimal integer n such that there is an exact sequence $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$ with all G_i 's Gorenstein projective, or ∞ if no such exact sequence exists. The Gorenstein injective dimension of M , denoted by $\text{Gid}M$, is defined dually. Moreover, the supresum of Gorenstein projective dimensions of all R -modules coincides with the supresum of Gorenstein injective dimensions of all R -modules, which is called the Gorenstein global dimension of R and is denoted by $\text{Ggd}R$ [5].

We have the following result for these Gorenstein homological dimensions.

Theorem 3.7 *Let R be a ring and (M, δ_M) be a differential module and n an integer.*

- (1) $\text{Gpd}(M, \delta_M) \leq n$ if and only if $\text{Gpd}M \leq n$.
- (2) $\text{Gid}(M, \delta_M) \leq n$ if and only if $\text{Gid}M \leq n$.
- (3) $\text{Ggd}\mathbf{Diff}(\mathbf{R}) \leq n$ if and only if $\text{Ggd}R \leq n$.

Proof. (1) Assume first $\text{Gpd}(M, \delta_M) \leq n$. Then there is an exact sequence of differential modules

$$0 \rightarrow (M_n, \delta_n) \rightarrow (P_{n-1} \oplus P_{n-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow \cdots \rightarrow (P_0 \oplus P_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow (M, \delta_M) \rightarrow 0$$

such that P_i 's are projective R -modules and (M_n, δ_n) is a Gorenstein projective differential module, see for instance [9]. Hence, we have an exact sequence of R -modules

$$0 \rightarrow M_n \rightarrow P_{n-1} \oplus P_{n-1} \rightarrow \cdots \rightarrow P_0 \oplus P_0 \rightarrow M \rightarrow 0.$$

Note that each $P_i \oplus P_i$ is projective and that M_n is Gorenstein projective by Theorem 3.6, so we have that $\text{Gpd}M \leq n$.

Now assume that $\text{Gpd}M \leq n$. Then we have an exact sequence of R -modules

$$0 \rightarrow L_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow M \rightarrow 0$$

such that L_n is Gorenstein projective and that each C_i is projective, see also [9]. By Lemma 3.3, we obtain an exact sequence of differential modules

$$0 \rightarrow (N_n, \delta_n) \rightarrow (Q_{n-1} \oplus Q_{n-1}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow \cdots \rightarrow (Q_0 \oplus Q_0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \rightarrow (M, \delta_M) \rightarrow 0,$$

such that $Q_i \simeq \bigoplus_{k=0}^k C_k$ and $N_n \simeq Q_{n-1} \oplus L_n$. Obviously, Q_i 's are projective and N_n is Gorenstein projective. Hence (N_n, δ_n) is a Gorenstein projective differential module. It follows that $\text{Gpd}(M, \delta_M) \leq n$.

(2) Dually to the proof of (1).

(3) By (1) and the definition of Gorenstein global dimension. □

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